Individual Competition

**Problem I.1.** Let \( n \in \mathbb{Z}_{>0} \) and let \( \mathcal{M}_n(\mathbb{R}) \) be the \( n \times n \) matrices with real entries. Let \( f : \mathcal{M}_n(\mathbb{R})^2 \to \mathcal{M}_n(\mathbb{R})^2 \) be the function \( f(A, B) = (AB, BA) \).

a. Is the function \( f \) injective?

b. Is the function \( f \) surjective?

c. Find all pairs \((A, B)\) of skew-symmetric real matrices which are a fixed point of \( f \).

**Problem I.2.** Let \((a_n)_n\) be a sequence of real numbers with \( a_0 > 0 \) and with

\[
a_{n+1} = \frac{a_n}{a_n^2 + a_n + 1} \quad \text{for all } n \in \mathbb{N}.
\]

a. Show that \( \lim_{n \to \infty} a_n = 0 \).

b. Determine \( \lim_{n \to \infty} na_n \).

**Problem I.3.** We have \( n \) coins, each of them gives the two outcomes head (H) and tail (T) after a toss, but the probabilities may be different for each coin.

We know that the probability of getting an even number of heads after a single toss of all coins is the same as getting an odd number of heads.

Must there be a fair coin among the \( n \) coins?

**Problem I.4.** Are there more positive integers with \( \sigma(\tau(n)) = n \) or more positive integers with \( \tau(\sigma(n)) = n \)?

(The function \( \tau(k) \) is the number of divisors of an integer \( k \) and the function \( \sigma(k) \) is the sum of divisors of an integer \( k \).)
Team Competition

Problem T.1. A graph is a set \( V \) of vertices together with a set \( E \) of edges. Each edge is a set of two distinct vertices, the endpoints of the edge. We represent the vertices by points in the plane and the vertices by lines connecting the two endpoints.

The distance \( \text{dist}(u, v) \) between two vertices \( v \) and \( w \) is the smallest number of edges we need to traverse to get from \( v \) to \( w \). For example, the distance of a vertex to a vertex it shares an edge with is 1, the distance of a vertex to itself is 0.

For a fixed natural number \( n \geq 2 \), we consider two graphs. The star \( S_n \) is the graph with vertices \( V = \{1, 2, \ldots, n\} \) and edges \( \{\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}\} \). The path is the graph with vertices \( V = \{1, 2, \ldots, n\} \) and edges \( E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\} \).

Show that

\[
\det_{1 \leq i, j \leq n} (\text{dist}(i, j))
\]

gives the same result for both graphs.

Problem T.2. Evaluate

\[
\int_0^\pi \frac{x \sin x}{\sqrt{1 + (\sin x)^2}} \, dx.
\]

Problem T.3. Let \( e_n \) be the number of subsets of \( \{1, 2, \ldots, 2n\} \) that contain more even than odd elements.

Determine all \( n \geq 1 \) such that \( e_n \) is odd.

Problem T.4. Let \( (a_n)_n \) be the sequence with \( a_1 = 2 \) and

\[
a_{n+1} = a_n^3 - a_n^2 + 1.
\]

a. Show that the sequence does not contain a perfect square.

b. Show that the sequence does not contain a perfect cube.

Problem T.5. For each \( Q \in \mathbb{R}[x] \), we define the vector space

\[
V_Q = \{ P \in \mathbb{R}[x] : \deg P \leq 2024 \text{ and } P(Q(a)) = P(a) \text{ for all } a \in \mathbb{R}\}.
\]

What are all the possible values of \( \dim V_Q \) as \( Q \) runs through all polynomials?

Problem T.6. On a blackboard, we have the numbers \( x_1, \ldots, x_n \) with \( 0 < x_i < 1/n \). In each step, we choose two numbers \( x \) and \( y \) and replace them with \( x \sqrt{1 - y^2} + y \sqrt{1 - x^2} \).

After \( n - 1 \) steps, we are left with a single number. Show that the number does not depend on which two numbers we choose in each step.
Problem T.7. Consider functions $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ such that $f(f(n)) \leq f(n + 1) - f(n)$ for all $n \in \mathbb{N}$.
What is the maximal value that $f(2024)$ can take?

Problem T.8. Let $s(x) = \min_{n \in \mathbb{Z}} |x - n|$ be the distance of $x$ to the nearest integer.
Show that the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n}$$

has a maximum.
Determine the maximum.