Individual Competition

Problem I.1. Let $n \in \mathbb{Z}_{>0}$ and let $\mathcal{M}_n(\mathbb{R})$ be the $n \times n$ matrices with real entries.
Let $f : \mathcal{M}_n(\mathbb{R})^2 \to \mathcal{M}_n(\mathbb{R})^2$ be the function $f(A, B) = (AB, BA)$.

a. Is the function f injective?

b. Is the function f surjective?

c. Find all pairs (A, B) of skew-symmetric real matrices which are a fixed point of f.

Problem I.2. Let $(a_n)_n$ be a sequence of real numbers with $a_0 > 0$ and with

$$a_{n+1} = \frac{a_n}{a_n^2 + a_n + 1} \text{ for all } n \in \mathbb{N}.$$

a. Show that $\lim_{n\to\infty} a_n = 0$.

b. Determine $\lim_{n\to\infty} na_n$.

Problem I.3. We have n coins, each of them gives the two outcomes head (H) and tail (T) after a toss, but the probabilities may be different for each coin.
We know that the probability of getting an even number of heads after a single toss of all coins is the same as getting an odd number of heads.
Must there be a fair coin among the n coins?

Problem I.4. Are there more positive integers with $\sigma(\tau(n)) = n$ or more positive integers with $\tau(\sigma(n)) = n$?
(The function $\tau(k)$ is the number of divisors of an integer k and the function $\sigma(k)$ is the sum of divisors of an integer k.)

Team Competition

Problem T.1. A graph is a set V of vertices together with a set E of edges. Each edge is a set of two distinct vertices, the endpoints of the edge. We represent the vertices by points in the plan and the vertices by lines connecting the two endpoints.
The distance $\operatorname{dist}(u,v)$ between two vertices v and w is the smallest number of edges we need to traverse to get from v to w. For example, the distance of a vertex to a vertex it shares an edge with is 1, the distance of a vertex to itself is 0.
For a fixed natural number $n \geq 2$, we consider two graphs. The star S_n is the graph with vertices $V = \{1, 2, \ldots, n\}$ and edges $\{\{1,2\}, \{1,3\}, \ldots, \{1,n\}\}$. The path is the graph with vertices $V = \{1, 2, \ldots, n\}$ and edges $E = \{\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}\}$.
Show that

$$\det_{1 \leq i,j \leq n} (\operatorname{dist}(i,j))$$

gives the same result for both graphs.

Problem T.2. Evaluate

$$\int_0^\pi \frac{x \sin x}{\sqrt{1 + (\sin x)^2}}\, dx.$$

Problem T.3. Let e_n be the number of subsets of $\{1, 2, \ldots, 2n\}$ that contain more even than odd elements.
Determine all $n \geq 1$ such that e_n is odd.

Problem T.4. Let $(a_n)_n$ be the sequence with $a_1 = 2$ and

$$a_{n+1} = a_n^3 - a_n^2 + 1.$$

a. Show that the sequence does not contain a perfect square.

b. Show that the sequence does not contain a perfect cube.

Problem T.5. For each $Q \in \mathbb{R}[x]$, we define the vector space

$$V_Q = \{P \in \mathbb{R}[x] : \deg P \leq 2024 \text{ and } P(Q(a)) = P(a) \text{ for all } a \in \mathbb{R}\}.$$

What are all the possible values of $\dim V_Q$ as Q runs through all polynomials?

Problem T.6. On a blackboard, we have the numbers $x_1, \ldots, x_n$ with $0 < x_i < 1/n$. In each step, we choose two numbers x and y and replace them with $x\sqrt{1-y^2} + y\sqrt{1-x^2}$.
After $n-1$ steps, we are left with a single number. Show that the number does not depend on which two numbers we choose in each step.

Problem T.7. Consider functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $f(f(n)) \leq f(n+1) - f(n)$ for all $n \in \mathbb{N}$.
What is the maximal value that $f(2024)$ can take?

Problem T.8. Let $s(x) = \min_{n\in\mathbb{Z}} |x - n|$ be the distance of x to the nearest integer.
Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n}$$

has a maximum.
Determine the maximum.