Individual Competition

Problem I.1. Let $n \in \mathbb{Z}_{>0}$ and let $\mathcal{M}_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Let $f : \mathcal{M}_n(\mathbb{R})^2 \to \mathcal{M}_n(\mathbb{R})^2$ be the function $f(A, B) = (AB, BA)$.

- **a.** Is the function f injective?
- **b.** Is the function f surjective?
- **c.** Find all pairs (A, B) of skew-symmetric real matrices which are a fixed point of f.

Solution to Problem I.1. a. The function is not injective because $f(0, I) = f(I, 0)$ where I is the identity matrix of size n and 0 is the zero matrix of size n.
b. The function is not surjective because $\operatorname{tr}(AB - BA) = 0$, therefore, the couple $(I, 0)$ is not in the image of f.
Alternatively, if $(AB, BA) = (I, 0)$, we get $AB = I$ and therefore, $BA = I \neq 0$.
c. We get

$$A^2 = (AB)A = A(BA) = AB = A,$$

and analogously $B^2 = B$. Therefore, A and B are diagonalizable matrices with eigenvalues 0 and 1, but skew-symmetric matrices have only imaginary eigenvalues different from zero. Therefore, $A = B = 0$ which is clearly a fixed point of skew-symmetric matrices.
Note that it is also possible to obtain and use

$$-A = A^T = (AB)^T = B^T A^T = (-B)(-A) = BA = B,$$

which makes it unnecessary to know the eigenvalues of skew-symmetric matrices.

Problem I.2. Let $(a_n)_{n\geq 0}$ be a sequence of real numbers with $a_0 > 0$ and with

$$a_{n+1} = \frac{a_n}{a_n^2 + a_n + 1} \text{ for all } n \in \mathbb{Z}_{\geq 0}.$$

- **a.** Show that $\lim_{n\to\infty} a_n = 0$.
- **b.** Determine $\lim_{n\to\infty} n a_n$.

Solution to Problem I.2. a. It is quite clear by induction that $a_n > 0$ for all n. This implies $a_{n+1} < a_n$. So we have a decreasing sequence of real numbers bounded below by 0. Therefore, it is convergent.

If we apply the limit to both side of the recurrence, we get

$$a = \frac{a}{a^2 + a + 1}$$

which gives $a = 0$ as desired.
b. Since a_n is a decreasing sequence, the sequence $\frac{1}{a_n}$ is increasing and we can apply Césaro-Stolz to

$$\frac{n}{\frac{1}{a_n}}$$

and get

$$\lim_{n\to\infty} n a_n = \lim_{n\to\infty} \frac{n}{\frac{1}{a_n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{a_{n+1}} - \frac{1}{a_n}}$$

provided the last limit exists. However, we can easily evaluate it by

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{\frac{1}{a_{n+1}} - \frac{1}{a_n}} &= \lim_{n\to\infty} \frac{a_n a_{n+1}}{a_n - a_{n+1}} \\
&= \lim_{n\to\infty} \frac{a_{n+1}(a_n^2 + a_n + 1)a_{n+1}}{a_{n+1}(a_n^2 + a_n + 1) - a_{n+1}} \\
&= \lim_{n\to\infty} \frac{(a_n^2 + a_n + 1)a_{n+1}}{a_n^2 + a_n} \\
&= \lim_{n\to\infty} \frac{a_n}{a_n^2 + a_n} \\
&= \lim_{n\to\infty} \frac{1}{a_n + 1} \\
&= 1.
\end{aligned}$$

Problem I.3. We have n coins, each of them having head (H) and tail (T) as possible outcomes after a toss, but the probabilities may be different for each coin.
We know that the probability of getting an even number of heads after tossing every coin exactly once is the same as getting an odd number of heads.
Must there be a fair coin among the n coins, i.e. a coin where head and tail come up with probability $\frac{1}{2}$ each?

Solution to Problem I.3. Let p_i the probability of getting H after a coin toss of coin i, and let $q_i = 1 - p_i$ be the probability of getting T with the same coin.

If we look at

$$\prod_{1\le i\le n}(q_i-p_i),$$

each term after multiplying out parentheses corresponds in absolute value to the appropriate probability of one of the outcomes for a single toss of all coins. However, the sign is positive if there is an even number of heads and negative if there is an odd number of heads.
Therefore, this product gives exactly the difference of probabilities of the two outcomes, which is zero by the problem statement.
If the product is zero, one of the factors must be zero, but $q_i = p_i$ means exactly that there is a fair coin.

Problem I.4. Let a be the number of positive integers with $\sigma(\tau(n)) = n$ and let b be the number of positive integers with $\tau(\sigma(n)) = n$.
Show that a and b are finite.
Do we have $a < b$, $b < a$ or $a = b$?
(The function $\tau(k)$ is the number of positive divisors of an integer k and the function $\sigma(k)$ is the sum of positive divisors of an integer k.)

Solution to Problem I.4. We note that $\tau(n) \le 2\sqrt{n}$ since divisors of n come in pairs with one number in the pair being at most $\sqrt{n}$.
Since the second largest divisor of n is at most $n/2$, we also have

$$\sigma(n) \le n + \sum_{k=1}^{n/2} k = n + \frac{n/2(n/2+1)}{2}.$$

Since both of these upper bounds are increasing functions of n, we immediately get

$$\sigma(\tau(n)) \le 2\sqrt{n} + \frac{\sqrt{n}(\sqrt{n}+1)}{2}$$

which is easily seen to be smaller than n for $n > 25$. and

$$\tau(\sigma(n)) \le 2\sqrt{n + \frac{n/2(n/2+1)}{2}}$$

which is easily seen to be smaller than n for $n > 10$.
Now we can either check the remaining cases or use the following argument to see that $a = b$:
For any n with $\sigma(\tau(n)) = n$, we set $k = \tau(n)$ and get $\tau(\sigma(k)) = \tau(n) = k$.
This gives an immediate bijection between the two sets of solutions and we get that $a = b$.
(We have $\sigma(\tau(n) = n$ for $n = 1, 3, 4, 12$ and $\tau(\sigma(n))$ for $n = 1, 2, 3, 6$.)

Team Competition

Problem T.1. A graph is a set V of vertices together with a set E of edges. Each edge is a set of two distinct vertices, the endpoints of the edge. We represent the vertices by points in the plane and the edges by lines connecting the two endpoints.
The distance $\mathrm{dist}(u, v)$ between two vertices u and v is the smallest number of edges we need to traverse to get from u to v. For example, the distance of a vertex to a vertex it shares an edge with is 1, the distance of a vertex to itself is 0.
For a fixed natural number $n \geq 2$, we consider two graphs. The star S_n is the graph with vertices $V = \{1, 2, \dots, n\}$ and edges $\{\{1, 2\}, \{1, 3\}, \dots, \{1, n\}\}$. The path P_n is the graph with vertices $V = \{1, 2, \dots, n\}$ and edges $E = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}\}$.
Show that

$$\det_{1\leq i,j\leq n}(\mathrm{dist}(i, j))$$

gives the same result for both graphs.

Solution to Problem T.1. First solution:

We calculate both determinants explicitly.
The star graph has the distances

$$\mathrm{dist}(i, j) = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } (i = 1 \text{ and } j \neq 1) \text{ or } (j = 1 \text{ and } i \neq 1), \\ 2 & \text{else.} \end{cases}$$

We multiply the first row with 2 and then we divide all columns except the first one by 2.
This shows that the determinant of the star graph is $2^{n-2} \det M$ where

$$M_{i,j} = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases}$$

This determinant can be evaluated in many ways. A quick way is to replace 0 by x, note that $x = 1$ gives a matrix of rank 1, and $x = -(n-1)$ gives a matrix of rank $n-1$. Since the determinant of the matrix is a monic polynomial in x of degree n, we get

$$\det M = (0-1)^{n-1}(0-(-(n-1)) = (-1)^{n-1}(n-1).$$

We conclude that the determinant of the distances of the star graph is

$$(-1)^{n-1}(n-1)2^{n-2}.$$

For the path, we get that

$$\operatorname{dist}(i,j) = |i-j|.$$

By replacing row_k with $\text{row}_k - \text{row}_{k+1}$ for $k = 1, 2, \ldots n-1$ and then column_k with $\text{column}_k - \text{column}_{k+1}$ for $k = 1, 2, \ldots n-1$, we obtain the matrix A with

$$A_{ij} = \begin{cases} -2 & \text{if } i = j \neq n, \\ 1 & \text{if } (i = n \text{ and } j = n-1) \text{ or } (j = n \text{ and } i = n-1), \\ 0 & \text{else.} \end{cases}$$

If we call the determinant of this matrix d_n, we see by expansion with respect to the first row that

$$d_n = (-2)d_{n-1} + (-1)^{n-1} \cdot 1 \cdot (-2)^{n-2}.$$

Since $d_2 = -1$, we can easily prove by induction that

$$d_n = (-1)^{n-1}(n-1)2^{n-2},$$

as desired.
Note that this is a special case of the result that the determinant of the distance matrix is the same for all trees.

Second solution:

We regard a connected graph G with vertices $1, \ldots, n$ where vertex ℓ is a leaf, i.e. a vertex with a single edge joining it. Let v be the neighboring vertex.
We see that the distance of ℓ to itself is 0, and any other distance $\operatorname{dist}(\ell, k) = \operatorname{dist}(v, k) + 1$, since any path from ℓ must pass through v.
This means that the row operation on the distance matrix that replaces row_ℓ with $\text{row}_\ell - \text{row}_v$ gives us the new first row $(-1, 1, \ldots, 1)$.
Now, we do the same for the columns and replace column_ℓ with $\text{column}_\ell - \text{column}_v$ and get first column $(-2, 1, \ldots, 1)$.
Since these operations do not change the determinant and the resulting matrix does not depend on which vertex v the leaf ℓ was attached to, we see that we can move leaves from one place to another without changing the determinant.
In particular, we can take the path P_n and starting from n, we move n from its neighbor $n-1$ to the neighbor 1. Then we take the leaf $n-1$ and move it to the neighbor 1, and so on, until we have the star.
Note that this proof works for all trees.

Problem T.2. Evaluate

$$\int_0^\pi \frac{x \sin x}{\sqrt{1 + (\sin x)^2}}\, dx.$$

Solution to Problem T.2. We call the integral I and note that by substituting x by $\pi - x$, we get

$$I = \int_0^\pi \frac{(\pi - x)\sin x}{\sqrt{1 + (\sin x)^2}}\, dx$$

and therefore,

$$2I = \int_0^\pi \frac{\pi \sin x}{\sqrt{1 + (\sin x)^2}}\, dx = \int_0^\pi \frac{\pi \sin x}{\sqrt{2 - (\cos x)^2}}\, dx = \pi \int_{-1}^1 \frac{1}{\sqrt{2 - t^2}}\, dt,$$

where the last equality comes from the substitution $t = \cos x$.
Now, we evaluate the remaining integral

$$\pi \int_{-1}^1 \frac{1}{\sqrt{2 - t^2}}\, dt = \frac{\pi}{\sqrt{2}} \int_{-1}^1 \frac{1}{\sqrt{1 - (t/\sqrt{2})^2}}\, dt = \frac{\pi}{\sqrt{2}} \int_{\pi/4}^{3\pi/4} \frac{\sqrt{2}\sin u}{\sqrt{1 - (\cos u)^2}}\, du = \pi \int_{\pi/4}^{3\pi/4} \frac{\sin u}{\sin u}\, du = \pi^2/2,$$

where we use the substitution $t = \sqrt{2}\cos u$.
Therefore, $I = \pi^2/4$.

Problem T.3. Let e_n be the number of subsets of $\{1, 2, \ldots, 2n\}$ that contain more even than odd elements.
Determine all $n \geq 1$ such that e_n is odd.

Solution to Problem T.3. First solution:

We have

$$2e_n = 2 \sum_{0 \leq k < l \leq n} \binom{n}{k}\binom{n}{l} = \sum_{0 \leq k, l \leq n} \binom{n}{k}\binom{n}{l} - \sum_{k=0}^{n} \binom{n}{k}^2 = 2^n \cdot 2^n - \binom{2n}{n},$$

using the binomial theorem and the Vandermonde identity.
Therefore, we have

$$e_n = 2^{2n-1} - \frac{1}{2}\binom{2n}{n} = 2^{2n-1} - \binom{2n-1}{n-1} \equiv \binom{2n-1}{n-1} \pmod{2}.$$

The theorem of Lucas tells us this is odd if and only if adding n and $n-1$ in the binary representation has no carries. This is only possible if the highest digit of n changes when 1 is subtracted which is the case exactly for powers of 2 who have the desired property.

Second solution:

Let $n = 2k + 1$ be odd.
We partition even and odd numbers up to n into k pairs and a remaining number, i.e.

$$(1, 3), (5, 7), \ldots, 2n - 1, (2, 4), (6, 8), \ldots, 2n.$$

The subsets containing 1, but not 3 are in bijection with the subsets containing 3, but not 1, since we just replace 1 with 3.
This bijection does not change the number of even and odd elements, so these subsets contribute an even number to e_n and do not change the parity of e_n.
Therefore, it is enough to consider subsets that contain both or none of 1 and 3.
This argument can be repeated for all the pairs, since the bijections also work on the restricted sets of subsets.
We conclude that it is enough to count the subsets where each of the $2k$ pairs occurs together or not at all. If there are more even pairs than odd pairs, the single odd number cannot change that there are more even numbers. Therefore, the single numbers can be freely chosen to be included or excluded which gives a factor of 4, and we have again an even contribution to e_n.
If there are less even than odd pairs, than the single even number cannot change that there are more odd than even numbers and there is no contribution to e_n.
The remaining case are the subsets with equally many even and odd pairs. To get a subset with more even numbers, we have to choose the single even number and exclude the single odd number. For each cardinality ℓ, we have to count the square of the number of ℓ-element subsets of k elements. However, we are only interested in the parity which does not change if we drop the square, so we just get the sum of all subsets of k elements which is 2^k, an even number for $k > 0$ and an odd number for $k = 0$.
Therefore, e_n is even for n odd with $n > 1$ while e_1 is odd.
Now, we consider even $n = 2k$.
As before, we partition even and odd numbers into pairs. However, this time there are no remaining numbers and we have k even pairs and k odd pairs. With the same argument as before, it is enough to consider subsets where each pair occurs together or not at all.
This gives for even n that $e_n \equiv e_{n/2} \mod 2$.
We conclude that for $n = 2^s u$ with u odd, we have $e_n \equiv e_u \mod 2$.
Therefore, e_n is odd if and only if n is a power of 2.

Problem T.4. Let $(a_n)_n$ be the sequence with $a_1 = 2$ and

$$a_{n+1} = a_n^3 - a_n^2 + 1.$$

a. Show that the sequence does not contain a perfect square.

b. Show that the sequence does not contain a perfect cube.

Solution to Problem T.4. **a.** If we look at the sequence modulo 3, we see that all elements of the sequence are 2 modulo 3 by induction. Therefore, they cannot be a square.

Alternatively, we can see that $a_{n+1} = (a_1 a_2 \cdot \cdots \cdot a_n)^2 + 1$ and therefore, cannot be a square, since the sequence is increasing and does not contain 0.

b. It is easy to see that a_{n+1} is between $(a_n - 1)^3$ and a_n^3, and therefore cannot be a cube.

Problem T.5. For each $Q \in \mathbb{R}[x]$, we define the vector space

$$V_Q = \{P \in \mathbb{R}[x] : \deg P \leq 2024 \text{ and } P(Q(a)) = P(a) \text{ for all } a \in \mathbb{R}\}.$$

What are all the possible values of $\dim V_Q$ as Q runs through all polynomials?

Solution to Problem T.5. If Q is constant, we obtain that P has to be constant, so $\dim V_Q = 1$. Constant P are clearly always elements of V_Q.
If $\deg P = d \geq 1$ and $\deg Q = e \geq 1$, we get $de = d$ which implies $e = 1$. It is enough to consider monic polynomials P and we see from the leading term that the leading coefficient of Q must be ± 1.
Case 1: $Q(x) = x$
Clearly, every P is a solution, so $\dim V_Q = 2025$.
Case 2: $Q(x) = x + c$ with $c \neq 0$.
Then we get $P(a) = P(a + c) = P(a + 2c) = \dots$. Since P takes the same value infinitely often, it has to be constant, so there are only the constant polynomials in V_Q in this case and the dimension is 1.
Case 3: $Q(x) = -x + c$
We get $P(-a + c) = P(a)$ for all a. Now set $x = a - c/2$ and get $P(-(x + c/2) + c) = P(x + c/2)$ for all x which gives $P(-x + c/2) = P(x + c/2)$.
This means that $R(x) = P(x + c/2)$ is an even polynomial, so only terms of even degree can occur.
The dimension is clearly $2024/2 + 1 = 1013$
Therefore, we get

$$\dim V_Q = \begin{cases} 1 & \text{if } Q \text{ constant or } Q(x) = x + c \text{ with } c \neq 0 \text{ or } \deg Q > 1, \\ 1013 & \text{if } Q(x) = -x + c \\ 2025 & \text{if } Q(x) = x. \end{cases}$$

Problem T.6. On a blackboard, we have the numbers $x_1, \dots, x_n$ with $0 < x_i < 1/n$. In each step, we choose two numbers a and b and replace them with the number $a\sqrt{1-b^2} + b\sqrt{1-a^2}$.
After $n-1$ steps, we are left with a single number. Show that the number does not depend on which two numbers we choose in each step.

Solution to Problem T.6. We see that for $a = \sin\alpha$ and $b = \sin\beta$ with $0 < \alpha, \beta < \pi/2$, the operation returns

$$a\sqrt{1-b^2} + b\sqrt{1-a^2} = \sin\alpha\cos\beta + \sin\beta\cos\alpha = \sin(\alpha+\beta).$$

If we set $x_i = \sin\alpha_i$, the last number is clearly $\sin(\alpha_1 + \alpha_2 + \cdots + \alpha_n)$, because the condition $x_i < 1/n$ implies that $\alpha_i < \pi/(2n)$, so all intermediate sums are smaller than $\pi/2$. One way to show the inequality is to use the inequality $\sin x > x - x^3/3!$ for $x > 0$.

Problem T.7. Consider functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $f(f(n)) \leq f(n+1) - f(n)$ for all $n \in \mathbb{Z}_{\geq 0}$.
What is the maximal value that $f(2024)$ can take?

Solution to Problem T.7. The function f is montonously increasing because $f(n+1) - f(n) \geq f(f(n)) \geq 0$. If the function f is the zero function, we have a solution.
From now on, we assume that there is a smallest n_0 with $f(n_0) > 0$.
Case 1: We have $f(n) < n_0$ for all $n \in \mathbb{N}$.
In particular, $n_0 > 0$ which implies $f(0) = 0$. Also, we always have $f(f(n)) = 0$ and the inequality is satisfied if and only if the function is monotonously increasing. All such functions are solutions.
Case 2: There exists an n_1 such that $f(n_1) \geq n_0$.
For $n \geq n_1$, we get $f(n+1) - f(n) \geq f(f(n)) \geq f(f(n_1)) \geq f(n_0) = c > 0$ which implies $f(n+1) \geq f(n) + c$. Therefore, the function f admits arbitrary large values and is monotonously increasing and both these properties are also true for $f(f(n))$ and we can choose n_1 such that $c > 1$.
This implies $f(n) \geq cn + d$ with $d = f(n_1) - cn_1 \in \mathbb{Z}$ for all $n \geq n_1$. But we also have $f(f(n)) \leq f(n+1) - f(n) < f(n+1)$ and by monotony $f(n) \leq n+1$. Therefore, we have $cn + d \leq n+1$ for sufficiently large n. But this inequality cannot be true for large n because of $c > 1$, so there are no solutions in this case.
In summary, the solutions are the zero function and all monotonously increasing functions with $f(0) = 0$ that are bounded by the last argument where they take the value 0.
We conclude that for $f(2024)$ to be larger than 0, the number 2023 is the largest possible last argument to be 0, and the function

$$f(n) = \begin{cases} 0 & \text{if } n \leq 2023, \\ 2023 & \text{if } n \geq 2024 \end{cases}$$

attains this maximum.

Problem T.8. Let $s(x) = \min_{n\in\mathbb{Z}} |x-n|$ be the distance of x to the nearest integer.
Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n}$$

has a maximum.
Determine the maximum.

Solution to Problem T.8. It is possible to show that the Takagi function is continuous by noting that it is a sum of continuous functions and converges uniformly.
However, this is not necessary because we can determine the maximum explicitly.
We are interested in combining two consecutive terms, therefore, we look at the function

$$t(x) = s(x) + \frac{s(2x)}{2} = \begin{cases} x + (2x)/2 = 2x & \text{if } x \in [0, 1/4] \\ x + (1-2x)/2 = 1/2 & \text{if } x \in [1/4, 2/4] \\ (1-x) + (2x-1)/2 = 1/2 & \text{if } x \in [2/4, 3/4] \\ (1-x) + (2-2x)/2 = 2(1-x) & \text{if } x \in [3/4, 4/4] \end{cases}$$

The maximum of $t(x)$ is clearly $1/2$, and

$$s(2^{2k}x)/2^{2k} + s(2^{2k+1}x)/2^{2k+1} = t(4^k x)/4^k$$

has the maximum $\frac{1}{2}\frac{1}{4^k}$.
Therefore,

$$f(x) \leq \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{4}} = \frac{1}{2}\cdot\frac{4}{3} = \frac{2}{3}.$$

The upper bound is attained if $4^k x \in [1/4, 3/4]$ for all k, but this just means choosing the digits of x in basis 4 to always be 1 or 2. For example

$$x = (0.1111111\ldots)_4 = \frac{1}{3}$$

attains the maximum, but there are uncountably many x where the maximum is attained.