Individual Competition

April 8, 2026

Problem I.1. Let n be a positive integer and C be the $n \times n$ matrix with entries

$$C_{k\ell} = \cos\left(\frac{(k+\ell)\pi}{n}\right) \quad \text{for } 0 \le k, \ell \le n-1.$$

Calculate $\det C$.

Problem I.2. Let $I_n = \int_1^e x^2 (\ln x)^n \, dx$. Determine the limit

$$\lim_{n\to\infty} n \cdot I_n.$$

Problem I.3. **a.** Hugh is repeatedly tossing a fair coin with sides Heads and Tails until he first sees the sequence HH (i. e., two Heads in a row), then he stops. What is the expected number of coin flips?

b. Harriet is also repeatedly tossing a fair coin with sides Heads and Tails, but she stops when she first sees the sequence HT (i. e., Heads followed by Tails). She notices that they both have the same probability of stopping after exactly n flips where n is her favorite number. What are the possibilities for Harriet's favorite number based on this information?

Problem I.4. For any set $A \subseteq \mathbb{R}^2$, let $\partial A = \overline{A} \cap \overline{\mathbb{R}^2 \setminus A}$ be the boundary of A where $\overline{X}$ denotes the closure of a set X in $\mathbb{R}^2$.
What is the maximal size of the set

$$\{\partial^n A : n \ge 0\} = \{A, \partial A, \partial\partial A, \dots\}?$$

Team Competition

April 9, 2026

Problem T.1. An $n \times n$ matrix A with real entries is called *interesting* if

$$A^2 = 2A^T - I \quad \text{and} \quad \operatorname{tr}(A) + \operatorname{tr}(A^2) = 0.$$

a. Find all positive integers n such that an interesting matrix of size $n \times n$ exists.

b. Find the possible determinants of A for each positive integer n for which interesting matrices of size $n \times n$ exist.

Problem T.2. Let $p \neq 13$ be an odd prime number. In the beginning, the pair (a, b) with $a, b \in \mathbb{Z}$ is written on a blackboard. Every morning, the current pair (x, y) is erased and replaced with $(2x - 3y, 3x + 2y)$. Let N be the number of distinct pairs that will appear on the blackboard throughout this process if we consider the pairs (x, y) modulo p.
Prove that

$$N \mid p^2 - 1.$$

Problem T.3. The 8 vertices of a cube are labelled with integers. A vertex is *balanced* if the integer at that vertex is the average of the integers at its neighboring vertices. (Neighbors are vertices connected by an edge of the cube.)

a. Prove that a cube with 8 balanced vertices has the same integer at each vertex.

b. What is the maximal number of balanced vertices if the integers are not all equal?

Problem T.4. Let $(A, *)$ be a finite set of size n with a binary operation $*$. The following three properties hold:

P1: $a * a = a$ for all $a \in A$.

P2: $a * b = b * a$ for all $a, b \in A$

P3: $a * (a * b) = b$ for all $a, b \in A$.

Show that n has the form $6k + 1$ or $6k + 3$ for some integer k.

Problem T.5. Determine the limit

$$\lim_{q\to 1^-}(1-q)\sum_{i=0}^{\infty}\frac{q^i}{1+q^{2i}}.$$

Problem T.6. Johanna and Daniel are making magic cookies. Johanna can double the number of existing cookies at any time and as often as she likes. After each doubling, Daniel may choose to eat one cookie.

- **a.** Determine every possible number of cookies that can be reached starting from one cookie.
- **b.** What are the possible starting numbers of cookies from which they can reach 1025 cookies eventually?
- **c.** Prove that there are infinitely many integers k such that 3^k cookies can be made from 7 starting cookies.

Problem T.7. Let (X, d) be a metric space with metric d. Let $\gamma : [0, \infty) \to X$ be a curve in X with the property that $d(\gamma(s), \gamma(t)) = |s - t|$ for $s, t \in [0, \infty)$. For all $x \in X$, define

$$d_\gamma(x, \infty) = \lim_{t\to\infty}\left(d(x, \gamma(t)) - t\right).$$

- **a.** Show that the limit exists for all $x \in X$ (so that $d_\gamma(x, \infty)$ is well-defined).
- **b.** Let $X = \mathbb{R}^2$ with d the usual Euclidean distance, and let $\gamma(t) = (t, 0)$. Find all points $x \in \mathbb{R}^2$ with $d_\gamma(x, \infty) = 0$.
- **c.** Let $X = C([0, 1], \mathbb{R})$, the set of continuous functions with metric d given by the supremum norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$, and let $\gamma(t) = t\cdot\mathrm{id}$ where $\mathrm{id}(s) = s$ for all $s \in [0, 1]$. Determine $d_\gamma(f, \infty)$ for any $f \in X$.

Problem T.8. A peak of a permutation $(\pi_1, \pi_2, \ldots, \pi_n)$ is a triple of consecutive entries with $\pi_{k-1} < \pi_k > \pi_{k+1}$ for some $2 \le k \le n - 1$.

- **a.** What is the expected number of peaks in a permutation of $\{1, 2, \ldots, n\}$ when all permutations are chosen with equal probability?

A permutation π that can be written as the square $\pi = \sigma^2$ of another permutation is called a *square permutation*.

- **b.** What is the expected number of peaks in a square permutation of $\{1, 2, \ldots, n\}$ when all square permutations are chosen with equal probability?